

# Effect of electromagnetic fields on the creation of scalar particles in a flat Robertson-Walker space-time

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The influence of electromagnetic fields on the creation of scalar particles from vacuum in a flat Robertson-Walker space-time is studied. The Klein Gordon equation with varying electric field and constant magnetic one is solved. The Bogoliubov transformation method is applied to calculate the pair creation probability and the number density of created particles. It is shown that the electric field amplifies the creation of scalar particles while the magnetic field minimizes it.

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## I. INTRODUCTION

It is widely known that strong electric field creates particle-antiparticle pairs from the vacuum. This effect has many important applications in modern physics from heavy nucleus to black holes [1]. Several decades ago, Schwinger studied pair creation effects in the context of gauge invariance and vacuum polarization [2]. It has been shown that the vacuum to vacuum transition amplitude can be expressed through an intermediate effective action,

$$\mathcal{A}(vac - vac) = \exp(iS_{eff}), \quad (1)$$

and the pair creation probability can be extracted from the imaginary part of this action

$$\mathcal{P}_{Creat.} = 1 - |\mathcal{A}(vac - vac)|^2 \simeq 2 \text{Im } S_{eff}. \quad (2)$$

The probability of pair creation from vacuum is calculated in the presence of some electromagnetic fields [3, 4] and it has been concluded that electric field produces scalar particles

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with probability

$$\mathcal{P}_{Creat.} = \frac{e^2 E^2}{8\pi^3} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2} \exp\left(-n\pi \frac{m^2}{eE}\right) \quad (3)$$

while the magnetic field and the plane wave do not create pairs.

However, in spite of the fact that constant magnetic field does not produce particles, the probability given in (3) modifies to be

$$\mathcal{P}_{Creat.} = \frac{e^2 EH}{8\pi^2} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \csc\left(n\pi \frac{H}{E}\right) \exp\left(-n\pi \frac{m^2}{eE}\right) \quad (4)$$

when a magnetic field is added to the electric one. Therefore a magnetic field influences significantly the creation of particles.

Particle-antiparticle pairs may be created also by gravitational fields. This phenomenon is a prediction of quantum field theory in curved space-time and its study requires a definition of a vacuum state for the quantum fields [5–10]. However, it is well-known that in arbitrary curved background, there is no absolute definition of the vacuum state and the concept of particles is not completely clear. From physical point of view this is because in quantum theory a particle cannot be localized to a region smaller than its de Broglie wavelength. When this wavelength is sufficiently large, the concept of particle becomes unclear. In expanding universe spontaneous creation of particles occurs because the vacuum state is unstable - e.g., the vacuum state defined in the remote past differs from the vacuum state in the remote future. The effect of particle creation has many applications in contemporary cosmology - e.g., it could have consequences for early universe cosmology and may play an important role in the exit from inflationary universe and in the cosmic evolution [11–17].

The purpose of this paper is to study the effect of electromagnetic fields on the creation of scalar particles from vacuum in a flat Robertson-Walker space-time with the use of canonical method based on Bogoliubov relation between "in" and "out" states. We know that electromagnetic fields are abundant in the universe during the initial stages of its formation and certainly these fields have some influences on particle creation. In this context, electromagnetic fields influence the cosmic evolution directly via Friedman equations and by their effect on the creation of particles. Motivated by the fact that there is no electric field in the present stage of the universe, we consider in this work a varying electric field which vanishes at  $t \rightarrow \infty$ . The study of particle creation with stationary electric field is valid only when the creation of particles is significant for a short time interval. This is due to the fact that

created particles has an inverse effect on the electric field [22]. We note that the creation of scalar particles in electromagnetic and gravitational fields is discussed in several situations [18–22].

The paper is organized as follows; At the beginning we introduce a scalar field propagating in Robertson-Walker space-time and we give a general method to study the creation of scalar particles in a such space-time. Next we consider a solvable example with a varying electric field and we give two sets of exact solutions for the Klein Gordon equation. Then by the use of the relation between these two sets we determine the probability of pair creation, the number density of created particles and the vacuum persistence. For a particular case where the universe behaves like a radiation dominated one we calculate the total pair production probability from the vacuum to vacuum transition amplitude and we show how electric field amplifies particle creation. Finally we consider the combination of electric and magnetic fields.

## II. GENERAL FORMALISM FOR PARTICLE CREATION

In order to study the phenomenon of particle creation in gravitational fields we have at our disposal several methods such as the adiabatic method [23–25], the Hamiltonian diagonalization technique [26–29], the Green function approach [30, 31], the Feynman path integral technique [32, 33], the semiclassical WKB approximation [10, 34–37] as well as the method based on vacuum to vacuum transition amplitude and Schwinger-like effective action [38, 39] and the "in" and "out" states formalism [40–43] that we shall use in this paper.

To begin let us consider a scalar matter field  $\Phi$  with mass  $m$  and charge  $e$  subjected to a gravitational field described by the metric  $g_{\mu\nu}$  and an electromagnetic field described by the vector  $A_\mu$ . The dynamics of this system is in general governed by the following Klein Gordon equation

$$\frac{1}{\sqrt{-g}} (i\partial_\mu - eA_\mu) [g^{\mu\nu} \sqrt{-g} (i\partial_\nu - eA_\nu) \Phi] - (m^2 + \zeta_c R) \Phi = 0. \quad (5)$$

where  $R$  is the Ricci scalar and  $\zeta_c$  is a numerical parameter (In conformal coupling  $\zeta_c = \frac{1}{6}$ ).

We consider, in this work, a flat Robertson-Walker space-time provided with a metric of the form

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2) \quad (6)$$

This metric can be written as

$$ds^2 = C(\eta) [d\eta^2 - dx^2 - dy^2 - dz^2] \quad (7)$$

where  $\eta$  is the conformal time  $\eta = \int dt/a(t)$  and  $C(\eta)$  is the new scale factor defined by  $C(\eta) = \tilde{a}^2(\eta) \equiv a^2[t(\eta)]$ .

We choose to work with conformal time  $\eta$  which is convenient to the present coupling and we consider the gauge  $A^\mu = (0, 0, 0, A_z(\eta))$ . If we introduce a new field  $\psi(x)$  so that

$$\Phi(x) = \frac{1}{\sqrt{C(\eta)}} \psi(x) = \frac{1}{\sqrt{C(\eta)}} \chi(\vec{x}) \varphi(\eta), \quad (8)$$

where  $\chi(\vec{x})$  has, in the case of flat space-time, the form of a plane wave  $\chi(\vec{x}) \sim \exp(i\vec{k} \cdot \vec{r})$ , we can obtain the simplified equation

$$\left[ \frac{d^2}{d\eta^2} + \omega^2(\eta) \right] \varphi(\eta) = 0, \quad (9)$$

with

$$\omega^2(\eta) = [k_z - eA_z(\eta)]^2 + k_\perp^2 + m^2 C(\eta). \quad (10)$$

We assume that the space-time is asymptotically Minkowskian and the potential  $A_z(\eta)$  is asymptotically constant when  $\eta \rightarrow \pm\infty$ . This choice is suitable for the problem of particles creation. Since  $\omega(\eta)$  in equation (10) satisfies the super-adiabatic condition,

$$\lim_{\eta \rightarrow \pm\infty} \frac{\dot{\omega}}{\omega^2} = 0 \quad (11)$$

there is two adiabatic vacuum states and, consequently, the particle production is well-defined.

In such a case the solutions of the Klein Gordon equation have the following asymptotic behavior

$$\varphi_{in}^\epsilon(\eta) = \exp(-i\epsilon\omega_{in}\eta) \quad (12)$$

$$\varphi_{out}^\epsilon(\eta) = \exp(-i\epsilon\omega_{out}\eta), \quad (13)$$

where  $\epsilon$  indicates the positive or the negative frequency mode ( $\epsilon = \pm 1$ ) and  $\omega_{out}$  and  $\omega_{in}$  are given by

$$\omega_{in}^{out} = \lim_{\eta \rightarrow \mp\infty} \omega(\eta) \quad (14)$$

Let us search for the "in" and "out" vacuum states. In the first stage we write the field operator in it's Fourier decomposition

$$\hat{\psi}(\vec{x}, \eta) = \frac{1}{\sqrt{2}} \int d^3k \left[ \varphi_k^*(\eta) \chi_k(\vec{x}) \hat{a}_k + \varphi_k(\eta) \chi_k^*(\vec{x}) \hat{b}_k^\dagger \right] \quad (15)$$

where, in canonical quantization formalism, the operators  $\hat{a}_k$  and  $\hat{b}_k$  satisfy the following commutation relation

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = [\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta(\vec{k} - \vec{k}'). \quad (16)$$

With the help of the normalization condition

$$\varphi_k^* \dot{\varphi}_k - \varphi_k \dot{\varphi}_k^* = 2i, \quad (17)$$

we can find without difficulties the following expression of the Hamiltonian associated with the scalar field system

$$H = \frac{1}{2} \int d^3k \left[ E_k(\eta) (\hat{a}_k \hat{a}_k^\dagger + \hat{b}_k^\dagger \hat{b}_k) + F_k^*(\eta) \hat{b}_k \hat{a}_k + F_k(\eta) \hat{a}_k^\dagger \hat{b}_k^\dagger \right] \quad (18)$$

where

$$E_k(\eta) = |\dot{\varphi}_k(\eta)|^2 + \omega_k^2(\eta) |\varphi_k(\eta)|^2 \quad (19)$$

$$F_k(\eta) = \dot{\varphi}_k^2(\eta) + \omega_k^2(\eta) \varphi_k^2(\eta). \quad (20)$$

Here we remark that  $H$  is not diagonal at any time. However, since the mode functions  $\varphi_k(\eta)$  have asymptotic behaviors (12) and (13) at  $\eta \rightarrow \mp\infty$ , we can see that  $F_k(\eta) = 0$  and  $H$  becomes diagonal at  $\eta \rightarrow \pm\infty$ . In this situation we can define two vacuum states  $|0_{in}\rangle$  and  $|0_{out}\rangle$ . The state  $|0_{in}\rangle$  is an initial quantum vacuum state in the remote past with respect to a static observer and  $|0_{out}\rangle$  is a final quantum vacuum state in the remote future with respect to the same observer. This gives some vacuum instability which leads to particle creation.

Since equation (9) is of second order there are only two independent solutions and all other solutions can be expressed in terms of these two independent ones. Here we want to find two sets of independent solutions so that the two functions  $\varphi_{in}^\pm$  of the first set behave like positive and negative energy states at  $\eta \rightarrow -\infty$  and the two functions  $\varphi_{out}^\pm$  of the second set behave like positive and negative energy states at  $\eta \rightarrow +\infty$ . The relation between these two sets or the so-called Bogoliubov transformation is

$$\varphi_{in}^+ = \alpha \varphi_{out}^+ + \beta \varphi_{out}^- \quad (21)$$

$$\varphi_{in}^- = \beta^* \varphi_{out}^+ + \alpha^* \varphi_{out}^-, \quad (22)$$

where the Bogoliubov coefficients  $\alpha$  and  $\beta$  satisfy the condition  $|\alpha|^2 - |\beta|^2 = 1$ . The relation between the creation and annihilation operators is then

$$a_{out} = \alpha a_{in} + \beta^* b_{in}^\dagger \quad (23)$$

$$b_{out}^\dagger = \beta a_{in} + \alpha^* b_{in}^\dagger. \quad (24)$$

For the process of particle creation the probability amplitude that we want to calculate is defined by

$$\mathcal{A} = \langle 0_{out} | a_{out} b_{out} | 0_{in} \rangle. \quad (25)$$

Taking into account that

$$b_{out} = \frac{1}{\alpha^*} b_{in} + \frac{\beta^*}{\alpha^*} a_{out}^\dagger \quad (26)$$

we obtain

$$\mathcal{A} = \langle 0_{out} | a_{out} b_{out} | 0_{in} \rangle = \frac{\beta^*}{\alpha^*} \langle 0_{out} | 0_{in} \rangle. \quad (27)$$

The probability to create a pair of particles in state  $k$  from vacuum is then

$$\mathcal{P}_k = \left| \frac{\beta^*}{\alpha^*} \right|^2. \quad (28)$$

Let  $\mathcal{C}_k$  to be the probability to have no pair creation in the state  $k$ . The quantity  $\mathcal{C}_k (\mathcal{P}_k)^n$  is then the probability to have only  $n$  pairs in the state  $k$ . We have

$$\mathcal{C}_k + \mathcal{C}_k \mathcal{P}_k + \mathcal{C}_k (\mathcal{P}_k)^2 + \mathcal{C}_k (\mathcal{P}_k)^3 + \dots = 1 \quad (29)$$

or simply

$$\mathcal{C}_k = 1 - \mathcal{P}_k. \quad (30)$$

Being aware of  $|\frac{\beta^*}{\alpha^*}|^2 + |\frac{1}{\alpha^*}|^2 = 1$ , we can find the vacuum persistence which reads

$$\mathcal{C}_k = \left| \frac{1}{\alpha^*} \right|^2. \quad (31)$$

Another important result is the number density of created particles

$$n(k) = \langle 0_{in} | a_{out}^+ a_{out} | 0_{in} \rangle = |\beta|^2 \quad (32)$$

The general technique for investigating the process of particle creation being demonstrated, let us give an explicit example where the Klein Gordon equation admits exact and analytic solutions.

### III. SOLVABLE MODEL WITH VARYING ELECTRIC FIELD

Particle creation in Robertson-Walker space-time has been much discussed and the pair creation probability and the number density of created particles have been derived for several forms of the scale factor describing different stages of the evolution of the universe. For the present work we choose for the scale factor the form

$$C(\eta) = a + b \tanh(\lambda\eta) + c \tanh^2(\lambda\eta) \quad (33)$$

where  $a$ ,  $b$  and  $c$  are positive parameters. We can see that this form is the generalization of various particular cases found in literature; When  $c = 0$ , we have a cosmological model with  $C(\eta) = a + b \tanh(\lambda\eta)$  which has been widely studied [44–46]. With a particular choice of parameters  $a$ ,  $b$  and  $c$  we get some models discussed in [47, 48]. In addition, this universe becomes a radiation dominated one when  $a = b = 0$ ,  $c = \frac{a_0^4}{4\lambda^2}$  and  $\lambda \rightarrow 0$ . We can also make connection with a Milne universe (i.e.  $a(t) = a_1 t$ .) when  $c = 0$ ,  $\lambda = a_1$ ,  $b = a = \frac{a_1^2}{2\varepsilon}$  by making the change  $\eta \rightarrow \eta + \frac{\ln \varepsilon}{2\lambda}$  and taking the limit  $\varepsilon \rightarrow 0$ .

We choose for the varying electric field the gauge

$$A_\mu = \frac{E_0}{\lambda} \tanh \lambda \eta \delta_{\mu 3}, \quad (34)$$

which describes the following electric field

$$\vec{E} = \frac{1}{C(\eta)} \frac{E_0}{\cosh^2 \lambda \eta} \vec{u}_z. \quad (35)$$

This field becomes the so-called Sauter field in the case of Minkowski space-time (i.e.  $C(\eta) = 1$  and  $\eta = t$ ) [49].

In such a case  $\omega_{out}$  and  $\omega_{in}$  are given by

$$\omega_{in} = \sqrt{k^2 + m^2 (a + \bar{c} - \bar{b})} \quad (36)$$

$$\omega_{out} = \sqrt{k^2 + m^2 (a + \bar{c} + \bar{b})}. \quad (37)$$

and the simplified Klein Gordon becomes

$$\left( \frac{d^2}{d\eta^2} + k^2 + m^2 (a + \bar{b} \tanh \lambda \eta + \bar{c} \tanh^2 \lambda \eta) \right) \varphi = 0 \quad (38)$$

with

$$\begin{aligned} \bar{b} &= b - \frac{2ek_z E_0}{\lambda m^2} \\ \bar{c} &= c + \left( \frac{eE_0}{\lambda m} \right)^2 \end{aligned} \quad (39)$$

Now in order to solve equation (38) we make the change  $\eta \rightarrow \xi$ , where

$$\xi = \frac{1 + \tanh(\lambda \eta)}{2}. \quad (40)$$

The resulting equation that takes the form

$$\begin{aligned} &\left[ \frac{\partial^2}{\partial \xi^2} + \left( \frac{1}{\xi} - \frac{1}{1-\xi} \right) \frac{\partial}{\partial \xi} + \left( \frac{\omega_{in}^2}{4\lambda^2} \frac{1}{\xi} - \frac{m^2 c}{\lambda^2} \right. \right. \\ &\left. \left. + \frac{\omega_{out}^2}{4\lambda^2} \frac{1}{(1-\xi)} \right) \frac{1}{\xi(1-\xi)} \right] \tilde{\varphi}(\xi) = 0 \end{aligned} \quad (41)$$

is a Riemann type equation [50]

$$\begin{aligned} &\left[ \frac{\partial^2}{\partial \xi^2} + \left( \frac{1 - \alpha_1 - \alpha'_1}{\xi} - \frac{1 - \alpha_3 - \alpha'_3}{1-\xi} \right) \frac{\partial}{\partial \xi} + \right. \\ &\left. \left( \frac{\alpha_1 \alpha'_1}{\xi} - \alpha_2 \alpha'_2 + \frac{\alpha_3 \alpha'_3}{1-\xi} \right) \frac{1}{\xi(1-\xi)} \right] \tilde{\varphi}(\xi) = 0 \end{aligned} \quad (42)$$

where

$$\begin{aligned} \alpha_1 &= -\alpha'_1 = i \frac{\omega_{in}}{2\lambda} \\ \alpha_3 &= -\alpha'_3 = i \frac{\omega_{in}}{2\lambda} \\ \alpha_2 &= 1 - \alpha'_2 = \frac{1}{2} + \sqrt{\frac{m^2 \bar{c}}{\lambda^2} - \frac{1}{4}}, \end{aligned} \quad (43)$$

with the condition  $\alpha_1 + \alpha'_1 + \alpha_2 + \alpha'_2 + \alpha_3 + \alpha'_3 = 1$ .



Following [50] we can find for equation (42) several sets of solutions that can be written in terms of hypergeometric functions. Taking into account the behavior of positive and negative energy states we can classify our two sets as follows; for the "in" states we have

$$\begin{aligned} \tilde{\varphi}_{in}^+(\xi) &= \frac{1}{\sqrt{2\omega_{in}}} \xi^{-i\frac{\omega_{in}}{2\lambda}} (1-\xi)^{i\frac{\omega_{out}}{2\lambda}} \\ &F\left(\frac{1}{2} + i\frac{\omega_-}{\lambda} + i\delta, \frac{1}{2} + i\frac{\omega_-}{\lambda} - i\delta; 1 - i\frac{\omega_{in}}{\lambda}; \xi\right) \end{aligned} \quad (44)$$

and

$$\begin{aligned} \tilde{\varphi}_{in}^-(\xi) &= \frac{1}{\sqrt{2\omega_{in}}} \xi^{i\frac{\omega_{in}}{2\lambda}} (1-\xi)^{-i\frac{\omega_{out}}{2\lambda}} \\ &F\left(\frac{1}{2} - i\frac{\omega_-}{\lambda} + i\delta, \frac{1}{2} - i\frac{\omega_-}{\lambda} - i\delta; 1 + i\frac{\omega_{in}}{\lambda}; \xi\right), \end{aligned} \quad (45)$$

with

$$\omega_{\pm} = \frac{\omega_{out} \pm \omega_{in}}{2} \quad (46)$$

and

$$\delta = \frac{1}{2} \sqrt{\frac{4m^2\bar{c}}{\lambda^2} - 1}. \quad (47)$$

The factors  $(2\omega_{in})^{-1/2}$  and  $(2\omega_{out})^{-1/2}$  are determined by the use of the normalization condition (17) which explains the conservation of the Klein Gordon particle current.

For the "out" states we have

$$\begin{aligned} \tilde{\varphi}_{out}^+(\xi) &= \frac{1}{\sqrt{2\omega_{out}}} \xi^{-i\frac{\omega_{in}}{2\lambda}} (1-\xi)^{i\frac{\omega_{out}}{2\lambda}} \\ &F\left(\frac{1}{2} + i\frac{\omega_-}{\lambda} + i\delta, \frac{1}{2} + i\frac{\omega_-}{\lambda} - i\delta; 1 + i\frac{\omega_{out}}{\lambda}; 1-\xi\right) \end{aligned} \quad (48)$$

and

$$\begin{aligned} \tilde{\varphi}_{out}^-(\xi) &= \frac{1}{\sqrt{2\omega_{out}}} \xi^{i\frac{\omega_{in}}{2\lambda}} (1-\xi)^{-i\frac{\omega_{out}}{2\lambda}} \\ &F\left(\frac{1}{2} - i\frac{\omega_-}{\lambda} + i\delta, \frac{1}{2} - i\frac{\omega_-}{\lambda} - i\delta; 1 - i\frac{\omega_{out}}{\lambda}; 1-\xi\right). \end{aligned} \quad (49)$$

Let us, now use the relation between "in" and "out" solutions to determine the probability of pair creation and the number density of created particles. By the use of the relation

between hypergeometric functions [50]

$$\begin{aligned}
F(u, v; w; \xi) &= \frac{\Gamma(w) \Gamma(w - v - u)}{\Gamma(w - u) \Gamma(w - v)} F(u, v; u + v - w + 1; 1 - \xi) \\
&\quad + (1 - \xi)^{w-u-v} \frac{\Gamma(\gamma) \Gamma(u + v - w)}{\Gamma(u) \Gamma(v)} \\
&\quad F(w - u, w - v; w - v - u + 1; 1 - \xi)
\end{aligned} \tag{50}$$

and the property

$$F(u, v; w; \xi) = (1 - \xi)^{w-u-v} F(w - u, w - v; w; \xi), \tag{51}$$

we obtain

$$\alpha = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma(1 - i\frac{\omega_{in}}{\lambda}) \Gamma(-i\frac{\omega_{out}}{\lambda})}{\Gamma(\frac{1}{2} - i\frac{\omega_{+}}{\lambda} - i\delta) \Gamma(\frac{1}{2} - i\frac{\omega_{+}}{\lambda} + i\delta)} \tag{52}$$

and

$$\beta = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma(1 - i\frac{\omega_{in}}{\lambda}) \Gamma(i\frac{\omega_{out}}{\lambda})}{\Gamma(\frac{1}{2} + i\frac{\omega_{-}}{\lambda} - i\delta) \Gamma(\frac{1}{2} + i\frac{\omega_{-}}{\lambda} + i\delta)}. \tag{53}$$

The probability to create one pair of particles from vacuum is then

$$\mathcal{P}_k = \left| \frac{\Gamma(\frac{1}{2} - i\frac{\omega_{+}}{\lambda} - i\delta) \Gamma(\frac{1}{2} - i\frac{\omega_{+}}{\lambda} + i\delta)}{\Gamma(\frac{1}{2} + i\frac{\omega_{-}}{\lambda} - i\delta) \Gamma(\frac{1}{2} + i\frac{\omega_{-}}{\lambda} + i\delta)} \right|^2. \tag{54}$$

Using the following properties of the Gamma functions [50]

$$\Gamma(z + 1) = z\Gamma(z), \tag{55}$$

$$|\Gamma(ix)|^2 = \frac{\pi}{x \sinh \pi x} \tag{56}$$

and

$$\left| \Gamma\left(\frac{1}{2} + ix\right) \right|^2 = \frac{\pi}{\cosh \pi x} \tag{57}$$

we arrive at

$$\mathcal{P}_k = \frac{\cosh(2\pi\frac{\omega_{-}}{\lambda}) + \cosh\left(\pi\sqrt{\frac{4m^2\bar{c}}{\lambda^2} - 1}\right)}{\cosh(2\pi\frac{\omega_{+}}{\lambda}) + \cosh\left(\pi\sqrt{\frac{4m^2\bar{c}}{\lambda^2} - 1}\right)}. \tag{58}$$

For the vacuum persistence we obtain

$$\mathcal{C}_k = \frac{\cosh(2\pi\frac{\omega_{+}}{\lambda}) - \cosh(2\pi\frac{\omega_{-}}{\lambda})}{\cosh(2\pi\frac{\omega_{+}}{\lambda}) + \cosh\left(\pi\sqrt{\frac{4m^2\bar{c}}{\lambda^2} - 1}\right)}. \tag{59}$$

A simple calculation gives for the number density of created particles

$$n(k) = \frac{\cosh\left(2\pi\frac{\omega_-}{\lambda}\right) + \cosh\left(\pi\sqrt{\frac{4m^2\bar{c}}{\lambda^2} - 1}\right)}{\cosh\left(2\pi\frac{\omega_{\pm}}{\lambda}\right) - \cosh\left(2\pi\frac{\omega_-}{\lambda}\right)}. \quad (60)$$

Here we note that the number density of created particles can be written as

$$n(k) = \frac{1}{\left|\frac{\alpha}{\beta}\right|^2 - 1}, \quad (61)$$

and for large frequencies  $n(k)$  becomes

$$n(k) = \frac{1}{\exp\left(\frac{2\pi}{\lambda}\omega_{in}\right) - 1} \quad (62)$$

which is a thermal Bose-Einstein distribution.

Let us note that when  $b = c = 0$  and  $a = 1$  and by taking the limit  $\lambda \rightarrow 0$ , we obtain the well-known result associated with the constant electric field in Minkowski space-time

$$\mathcal{P}_k = \frac{\exp\left(-\pi\frac{k_{\perp}^2 + m^2}{eE_0}\right)}{1 + \exp\left(-\pi\frac{k_{\perp}^2 + m^2}{eE_0}\right)}, \quad (63)$$

with  $k_{\perp}^2 = k_x^2 + k_y^2$ .

#### IV. PARTICULAR CASE

Now we consider a cosmological model with a scale factor of the form

$$C(\eta) = a + \frac{a_0^4}{4}\eta^2 \quad (64)$$

which describes a radiation dominated like universe  $a(t) \sim a_0\sqrt{t}$ . It is obvious that this situation can be obtained by considering the particular case when  $b = 0$ ,  $c = \frac{a_0^4}{4\lambda^2}$  and by taking the limit  $\lambda \rightarrow 0$ . Here the role of the parameter  $a$  is to check the correctness of our results by making comparison to the case of Minkowski space-time when  $a = 1$  and  $a_0 = 0$ . To consider the particle creation in pure radiation dominated universe we have to put  $a = 0$ .

It is easy to show that when the scale factor is given by equation (64), the probability  $\mathcal{P}_k$  can be written in the form

$$\mathcal{P}_k = \frac{\sigma}{1 + \sigma}, \quad (65)$$

where

$$\sigma = \exp \left[ -2\pi \left( \frac{k_{\perp}^2 + am^2}{\sqrt{m^2 a_0^4 + 4e^2 E_0^2}} + \frac{m^2 a_0^4 k_z^2}{(m^2 a_0^4 + 4e^2 E_0^2)^{\frac{3}{2}}} \right) \right]. \quad (66)$$

The vacuum to vacuum transition probability is then

$$\exp(-2 \operatorname{Im} S_{eff}) = \prod_k \mathcal{C}_k = \prod_k \exp[-\ln(1 + \sigma)] \quad (67)$$

and consequently

$$2 \operatorname{Im} S_{eff} = \sum_k \ln(1 + \sigma). \quad (68)$$

Expanding the quantity  $\ln(1 + \sigma)$  and replacing the summation over  $k$  by  $\int \frac{d^3 k}{(2\pi)^3}$ , we get

$$2 \operatorname{Im} S_{eff} = \int \frac{d^3 k}{(2\pi)^3} \sum_{n=1} \frac{(-1)^{n+1}}{n} \exp \left[ -2n\pi \left( \frac{k_{\perp}^2 + am^2}{\sqrt{m^2 a_0^4 + 4e^2 E_0^2}} + \frac{m^2 a_0^4 k_z^2}{(m^2 a_0^4 + 4e^2 E_0^2)^{\frac{3}{2}}} \right) \right]. \quad (69)$$

By doing integration over  $k_x$  and  $k_y$  we obtain

$$2 \operatorname{Im} S_{eff} = \frac{\sqrt{m^2 a_0^4 + 4e^2 E_0^2}}{2(2\pi)^3} \sum_{n=1} \frac{(-1)^{n+1}}{n^2} \exp \left[ -2n\pi \left( \frac{am^2}{\sqrt{m^2 a_0^4 + 4e^2 E_0^2}} \right) \right] \int dk_z \exp \left[ -2n\pi \left( \frac{m^2 a_0^4}{(m^2 a_0^4 + 4e^2 E_0^2)^{\frac{3}{2}}} \right) k_z^2 \right]. \quad (70)$$

For the integration over  $k_z$  we can use the following property

$$dk_z = \frac{m^2 a_0^4 + 4e^2 E_0^2}{4eE_0} d\eta \quad (71)$$

to write  $2 \operatorname{Im} S_{eff}$  in the form

$$2 \operatorname{Im} S_{eff} = \int d\eta \Gamma(\eta) \quad (72)$$

where the particle creation probability per unit of time  $\Gamma(\eta)$  is given by

$$\Gamma(\eta) = \frac{(2e\mathcal{E})^3}{8(2\pi)^3 eE_0} \sum_{n=1} \frac{(-1)^{n+1}}{n^2} \exp \left[ -n\pi \left( \frac{am^2}{e\mathcal{E}} \right) \right] \exp \left[ -2n\pi \left( \frac{m^2 a_0^4}{16e^2 E_0^2} 2e\mathcal{E} \right) \eta^2 \right]. \quad (73)$$

with

$$4e^2\mathcal{E}^2 = m^2a_0^4 + 4e^2E_0^2. \quad (74)$$

Here we note that we obtain the Schwinger result by setting  $a = 1$  and  $a_0 = 0$ .

For  $a_0 \neq 0$ , by doing integration over conformal time  $\eta$ , we get a Schwinger-like series

$$2 \operatorname{Im} S_{eff} = \frac{1}{4\pi^3} \frac{(e\mathcal{E})^{\frac{5}{2}}}{ma_0^2} \sum_{n=1} \frac{(-1)^{n+1}}{n^{\frac{5}{2}}} \exp \left[ -n\pi \frac{m^2a}{e\mathcal{E}} \right] \quad (75)$$

Now if we consider a pure radiation dominated universe (i.e.  $a = 0$ ) we can see that the electric field amplifies the gravitational particle creation by the following factor

$$\gamma = \frac{2 \operatorname{Im} S_{eff}}{2 \operatorname{Im} S_{eff}(E_0 = 0)} = \left( 1 + 4 \frac{e^2 E_0^2}{m^2 a_0^4} \right)^{\frac{5}{4}}. \quad (76)$$

which is, for strong field, of order  $(eE_0/ma_0^2)^{\frac{5}{2}}$ .

For the number density of created particles we have

$$n(k) = \exp \left[ -2\pi \left( \frac{k_{\perp}^2}{\sqrt{m^2 a_0^4 + 4e^2 E_0^2}} + \frac{m^2 a_0^4 k_z^2}{(m^2 a_0^4 + 4e^2 E_0^2)^{\frac{3}{2}}} \right) \right]. \quad (77)$$

The total number of created particles can be written then in the form

$$N_T = \frac{(m^2 a_0^4 + 4e^2 E_0^2)^{\frac{3}{2}}}{8(2\pi)^3 e E_0} \int d\eta \exp \left[ -\pi \left( \frac{m^2 a_0^4}{8e^2 E_0^2} \sqrt{m^2 a_0^4 + 4e^2 E_0^2} \right) \eta^2 \right]. \quad (78)$$

Consequently we have

$$\frac{dN}{d\eta} = n(\eta) = \frac{(m^2 a_0^4 + 4e^2 E_0^2)^{\frac{3}{2}}}{8(2\pi)^3 e E_0} \exp \left[ -\pi \left( \frac{m^2 a_0^4}{8e^2 E_0^2} \sqrt{m^2 a_0^4 + 4e^2 E_0^2} \right) \eta^2 \right]. \quad (79)$$

By doing integration over  $\eta$  we get

$$N_T = \frac{(m^2 a_0^4 + 4e^2 E_0^2)^{\frac{5}{4}}}{\sqrt{8} (2\pi)^3 m a_0^2}. \quad (80)$$

We see that the factor  $\gamma$  can be obtained also from  $N_T$

$$\frac{N_T}{N_T(E_0 = 0)} = \gamma, \quad (81)$$

This effect seems to be important for light particles. However, this is not true. Since  $n(\eta)$  in equation (79) is Gaussian with respect to  $\eta$  we find that particle creation is significant in the time interval

$$\Delta\eta = \frac{1}{\sqrt{2\pi}} \frac{4eE_0}{ma_0^2} (m^2 a_0^4 + 4e^2 E_0^2)^{-\frac{1}{4}}. \quad (82)$$

In radiation dominated universe, electromagnetic backgrounds can be considered quasi-stationary only for short time  $\Delta\eta \ll 1$ . This gives

$$ma_0^2 \gg \sqrt{eE_0}. \quad (83)$$

The effect of electric field is then more important when the mass of created particles verifies the condition  $\sqrt{eE_0} \ll ma_0^2 \ll eE_0$ . Thus, electric field predominantly produces heavy particles. Furthermore, it is possible to create super-heavy particles with the mass of the Grand Unification scale in the early universe by strong electric field. This may have many important cosmological consequences. For light particles the effect of the electric field on particle creation is negligible although the factor  $\gamma$  becomes large when  $m$  is small. This is explained by the fact that, when  $m$  decreases  $N_T(E_0 = 0)$  decreases so that  $N_T = \gamma N_T(E_0 = 0)$  remains negligible.

## V. EFFECT OF MAGNETIC FIELD

Having studied the phenomenon of particle creation in the presence of an electric field, let us now consider the superposition of an electric field and a magnetic one to investigate the influence of the magnetic fields on the creation of scalar particles. For this aim we choose the gauge

$$A_\mu = \frac{E_0}{\lambda} \tanh(\lambda\eta) \delta_{\mu 3} - Hx \delta_{\mu 2} \quad (84)$$

which leads to the following Klein Gordon equation

$$\left[ \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial x^2} + \left( i \frac{\partial}{\partial y} + eHx \right)^2 + \left( i \frac{\partial}{\partial z} - \frac{eE_0}{\lambda} \tanh \lambda\eta \right)^2 + m^2 C(\eta) \right] \psi(\eta, \vec{x}) = 0. \quad (85)$$

To solve this equation we decompose  $\psi(\eta, \vec{x})$  as

$$\psi(\eta, \vec{x}) = \varphi(\eta)g(x) \exp(-ik_y y - ik_z z), \quad (86)$$

where the functions  $\varphi(\eta)$  and  $g(x)$  obey respectively

$$\left[ \frac{d^2}{d\eta^2} + \left( k_z - \frac{eE_0}{\lambda} \tanh \lambda\eta \right)^2 + m^2 C(\eta) + \kappa \right] \varphi(\eta) = 0 \quad (87)$$

and

$$\left[ -\frac{d^2}{dx^2} + (k_y + eHx)^2 \right] g(x) = \kappa g(x) \quad (88)$$

where  $\kappa$  is a constant resulting from the separation of variables. It is clear that by making the change  $x \rightarrow x - \frac{k_y}{eH}$ , equation (88) becomes similar to the wave equation associated with the harmonic oscillator, where the solution is given by

$$g(x) = \left( \frac{1}{2^l l! \sqrt{\pi}} \right)^{\frac{1}{2}} \left( \frac{eH}{4} \right)^{\frac{1}{2}} \exp \left[ -\frac{eH}{2} \left( x + \frac{k_y}{eH} \right)^2 \right] \mathcal{H}_l \left[ \sqrt{eH} \left( x + \frac{k_y}{eH} \right) \right] \quad (89)$$

and

$$\kappa = eH (2l + 1). \quad (90)$$

Here,  $l$  is an integer and  $\mathcal{H}_l(x)$  is the Hermit polynomial. For the function  $\varphi(\eta)$  we have

$$\left[ \frac{d^2}{d\eta^2} + k_z^2 + m^2 (a' + \bar{b} \tanh \lambda \eta + \bar{c} \tanh^2 \lambda \eta) \right] \varphi(\eta) = 0 \quad (91)$$

where

$$a' = a + \frac{eH}{m^2} (2l + 1) \quad (92)$$

The later equation is similar to (38) with the change  $a \rightarrow a'$ . Then with the same steps as in section (III) we can obtain the following results

$$\mathcal{P}_{k,l} = \frac{\cosh \left( 2\pi \frac{\omega'_-}{\lambda} \right) + \cosh (2\pi \delta)}{\cosh \left( 2\pi \frac{\omega'_+}{\lambda} \right) + \cosh (2\pi \delta)}, \quad (93)$$

where

$$\omega'_{in} = \sqrt{k_z^2 + m^2 (a' + \bar{c} - \bar{b})} \quad (94)$$

$$\omega'_{out} = \sqrt{k_z^2 + m^2 (a' + \bar{c} + \bar{b})}. \quad (95)$$

Let us remark here that when  $a = 1$ ,  $b = c = 0$  and  $\lambda \rightarrow 0$  we obtain the probability of pair creation in Minkowski space-time with electric and magnetic fields

$$\mathcal{P}_{k,l} = \frac{\exp \left( -\pi \frac{m^2 + eH(2l+1)}{eE_0} \right)}{1 + \exp \left( -\pi \frac{m^2 + eH(2l+1)}{eE_0} \right)}. \quad (96)$$

Like in the previous section when  $b = 0$ ,  $c = \frac{c_0}{\lambda^2}$  and  $\lambda \rightarrow 0$  we can get easily

$$2 \text{Im } S_{eff} = \int \frac{dk_y dk_z}{(2\pi)^2} \sum_l \sum_{n=1} \frac{(-1)^{n+1}}{n} \times \exp \left[ -n\pi \left( \frac{eH(2l+1) + am^2}{e\mathcal{E}} + \frac{m^2 a_0^4 k_z^2}{4(e\mathcal{E})^3} \right) \right]. \quad (97)$$

Taking into account that

$$dk_y = eHdx \quad (98)$$

we can write  $2 \text{Im } S_{eff}$  as follows

$$2 \text{Im } S_{eff} = \int dx d\eta \Gamma(x, \eta), \quad (99)$$

where  $\Gamma(x, \eta)$  is the pair creation probability per unit of time per unit of volume. It is given by

$$\begin{aligned} \Gamma(x, \eta) = & \frac{eH}{(2\pi)^2} \frac{e^2 \mathcal{E}^2}{eE_0} \sum_{n=1} \frac{(-1)^{n+1}}{n} \sum_l \exp \left[ -n\pi \frac{H}{\mathcal{E}} (2l+1) \right] \times \\ & \exp \left[ -n\pi \left( \frac{am^2}{e\mathcal{E}} + \frac{m^2 a_0^4 e\mathcal{E}}{4e^2 E_0^2} \eta^2 \right) \right]. \end{aligned} \quad (100)$$

Note here that  $\Gamma(x, \eta)$  does not depend on  $x$  because the universe is homogeneous and the magnetic field too.

By summing over  $l$

$$\sum_{l=0} \exp \left[ -n\pi \frac{H}{\mathcal{E}} (2l+1) \right] = \frac{1}{2 \sinh \left( n\pi \frac{H}{\mathcal{E}} \right)} \quad (101)$$

we get the Schwinger-like series

$$\Gamma(\eta) = \frac{1}{(2\pi)^3} \frac{(e\mathcal{E})^3}{eE_0} \sum_{n=1} \frac{(-1)^{n+1}}{n^2} F_n(H) \exp \left[ -n\pi \left( \frac{am^2}{e\mathcal{E}} + \frac{m^2 a_0^4 e\mathcal{E}}{4e^2 E_0^2} \eta^2 \right) \right], \quad (102)$$

where the factor  $F_n(H)$  that describes the effect of the magnetic field is given by

$$F_n(H) = \frac{n\pi \frac{H}{\mathcal{E}}}{\sinh \left( n\pi \frac{H}{\mathcal{E}} \right)}. \quad (103)$$

We remark that  $0 < F_n(H) \leq 1$ . This means that the magnetic field minimizes the creation of scalar particles.

For the number density we obtain

$$N_T(H, E_0) = \int d\eta \frac{\pi \frac{H}{\mathcal{E}}}{\sinh \left( \pi \frac{H}{\mathcal{E}} \right)} \frac{4e^2 \mathcal{E}^3}{4(2\pi)^3 eE_0} \exp \left[ -\pi \left( \frac{am^2}{e\mathcal{E}} + \frac{m^2 a_0^4 e\mathcal{E}}{4e^2 E_0^2} \eta^2 \right) \right]. \quad (104)$$

In the radiation era (i.e.  $a = 0$ ), we have

$$n(\eta) = \frac{\pi \frac{H}{\mathcal{E}}}{\sinh \left( \pi \frac{H}{\mathcal{E}} \right)} \frac{4e^2 \mathcal{E}^3}{4(2\pi)^3 eE_0} \exp \left[ -\pi \frac{m^2 e\mathcal{E}}{e^2 E_0^2} C(\eta) \right]. \quad (105)$$



This result is in complete agreement with equation (5.8) in [20].

Now by doing integration over  $\eta$  we obtain the expression

$$N_T = \left(1 + \frac{4e^2 E_0^2}{m^2 a_0^4}\right)^{\frac{5}{4}} \frac{\pi \frac{H}{\varepsilon}}{\sinh\left(\pi \frac{H}{\varepsilon}\right)} \frac{(ma_0^2)^{\frac{3}{2}}}{2\sqrt{2}(2\pi)^3}, \quad (106)$$

which reduces in the case of a pure magnetic field to

$$N_T(H, E_0 = 0) = \frac{\sqrt{ma_0^2}}{8\sqrt{2}\pi^2} \frac{eH}{\sinh\left(2\pi \frac{eH}{ma_0^2}\right)}. \quad (107)$$

Thus the magnetic field minimizes the gravitational particle creation by the factor

$$\gamma' = \frac{2\pi \frac{eH}{ma_0^2}}{\sinh\left(2\pi \frac{eH}{ma_0^2}\right)} < 1. \quad (108)$$

In the presence of both fields the amplification factor will be given by

$$\gamma'' = \frac{\pi Y (1 + X^2)^{\frac{3}{4}}}{\sinh\left(\pi \frac{Y}{\sqrt{1+X^2}}\right)}, \quad (109)$$

where  $X = \frac{2eE_0}{ma_0^2}$  and  $Y = \frac{2eH}{ma_0^2}$ . In this case  $\gamma''$  may be less than 1. This depends on the values of  $E_0$  and  $H$ . When  $H \sim E_0 \gg ma_0^2$ , we can see that  $\gamma'' \gg 1$  and the creation of super-heavy particles increases.

In addition, it is well-known that pure gravitational fields do not create massless particles with conformal coupling. Unlike results of reference [21] this remains true even if electromagnetic fields are present. In effect from equation (106) we can see that  $N_T = 0$  when  $m = 0$ .

## VI. CONCLUSION

In this paper we have studied the effect of electromagnetic fields on the creation of scalar particles in a Robertson-Walker space-time by considering the canonical method based on Bogoliubov transformation. We have given two sets of exact solutions for the Klein Gordon field equation with varying electric field and we have used these solutions to calculate the probability of pair creation and the number density of created particles.

Then we have discussed a particular cosmological model that behaves like radiation dominated universe where we have calculated the vacuum to vacuum transition probability and

we have extracted the nonvanishing imaginary term of the effective action that means that created particles are real and not virtual ones. We have considered also the combination of varying electric field and constant homogenous magnetic field.

The essential result is that strong electric field amplifies gravitational particle creation by a factor of order  $(eE_0/ma_0^2)^{\frac{5}{2}}$ . This conclusion is in agreement with the result of [22]. We have shown also that the magnetic field minimizes the particle creation like in case of Minkowski space-time with pure electromagnetic fields. Then the effect of electric field is more important than the magnetic field one vis-a-vis the process of cosmological scalar particle creation.

It is obvious that the inclusion of magnetic field may be done by making the change  $k_{\perp}^2 \rightarrow (2n+1)eH$ . This explains why the magnetic field minimizes the particles creation - e.g., since particles prefer to be created in lower energy state [4] and the minimum of  $k_{\perp}^2$  is 0 while the minimum of  $(2n+1)eH$  is  $eH$ .

In addition, the creation of massless particles with conformal coupling is impossible even if electromagnetic fields are present.

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- [1] R. Ruffini, G. Vereshchagin, S-S. Xue, Phys. Rep. **487** (2010)
  - [2] J. Schwinger, Phys. Rev. **82**, 664 (1951)
  - [3] E. Bresin and C. Itzykson, Phys. Rev. D **2**, 1191 (1970).
  - [4] E.S. Fradkin, D.M. Gitman and S.M. Shvartsman, Quantum Electrodynamics with Unstable Vacuum (Springer-Verlag, Berlin 1991)
  - [5] N.D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge Univ. Press, Cambridge 1982).
  - [6] S. A. Fulling, Aspects of Quantum Field Theory in Curved Space-Time (Cambridge University Press, Cambridge 1985)

- [7] V. F. Mukhanov and S. Winitzki; Introduction to Quantum Effects in Gravity (Cambridge Univ. Press, Cambridge, 2007)
- [8] A. A.Grib, S. G. Mamayev, and V.M.Mostepanenko,Vacuum Quantum Effects in Strong Fields ( Friedmann Lab. Publ.,St. Petersburg 1994)
- [9] L. Parker and D. J. Toms Quantum Field Theory in Curved Space-time: Quantized Fields and Gravity (Cambridge University Press, Cambridge 2009)
- [10] S. Winitzki; Phys. Rev. D **72**, 104011 (2005)
- [11] S. Debnath and A. K. Sanyal; Class. Quantum Grav. **28**, 145015 (2011)
- [12] J. A. S. Lima, F. E. Silva and R. C. Santos, Class. Quantum. Grav. **25**, 205006 (2008).
- [13] V. Sahni and S. Habib, Phys. Rev. Lett. **81**, 1766 (1998)
- [14] L. Kofman: Preheating After Inflation, Lect. Notes Phys. **738**, 55 (2008) (Springer-Verlag Berlin Heidelberg 2008)
- [15] J. Martin: Inflationary Perturbations: The Cosmological Schwinger Effect, Lect. Notes Phys.**738**, 193 (2008) (Springer-Verlag Berlin Heidelberg 2008)
- [16] I. Antoniadis, P. O. Mazur and E.Mottola, New J. Phys. **9**, 11 (2007)
- [17] J. Chen, P. Wu, H. Yu, Z. Li, Eur. Phys. J. C **72**, 1861 (2012)
- [18] V. M. Villalba, Phys. Rev. D **60**, 127501 (1999)
- [19] S. Haouat and R. Chekireb, Int. J. Theor. Phys. **51**, 1704 (2012)
- [20] G. Schäfer et H. Dehnen : J. Phys. A: Math. Gen, **13**, 517 (1980).
- [21] S. Moradi; Mod. Phys. Lett. A **24**, 1129 (2009).
- [22] I L Buchbinder and S D Odintsov, Sov. Phys. J. **25**, 385 (1982)
- [23] L.Parker, Phys. Rev. Lett. **21**, 562 (1968).
- [24] L.Parker, Phys.Rev. D **183**, 1057 (1969).
- [25] L.Parker, Phys.Rev. D **3**, 346 (1971).
- [26] A. A. Grip, S. G. Mamayev and V. M. Mostepanenko : Gen. Rel. Grav. **7**, 535 (1976).
- [27] A. A. Grip, S. G. Mamayev, and V. M. Mostepanenko, J. Phys. A: Math. Gen **13**, 2057 (1980)
- [28] J. Haro and E. Elizald; J.Phys. A: Math. Theor. **41**, 372003 (2008).
- [29] J. Haro;J.Phys. A: Math. Theor. **44** (2011).
- [30] S. Gavrilov, D.M. Gitman, and S.D. Odintsov, Int. J. Mod. Phys. A **12**, 4837 (1997)
- [31] I.L. Bukhbinder, Izv. Vyssh. Uchebn. Zaved. Fiz. **7**, 3 (1980).
- [32] I. H. Duru, N. Ünal : Phys. Rev. D **34**, 966 (1986).

- [33] D. M. Chitre and J. B. Hartle, Phys. Rev D. **16**, 251 (1977).
- [34] S. Biswas, J. Guha and N. G. Sarkar; Class. Quantum Grav. **12**, 1591 (1995)
- [35] J. Guha, D. Biswas, N. G. Sarkar and S. Biswas; Class. Quantum Grav. **12**, 1641 (1995)
- [36] S. Biswas, A. Shaw and P. Misra; Gen. Rel. Grav. **34**, 665 (2002)
- [37] S. Biswas and I. Chowdhury; Int. J. Mod. Phys. D **15**, 937 (2006)
- [38] E. Akhmedov; Mod. Phys. Lett. A **25**, 2815 (2010)
- [39] S. P. Kim; arXiv:1008.0577v1 [hep-th]
- [40] J. Garriga; Phys. Rev. D **49**, 6343 (1994)
- [41] V. M. Villalba and W. Greiner; Phys. Rev. D **65**, 025007 (2001)
- [42] V. M. Villalba; Progress of Theoretical Physics, **90**, 851 (1993).
- [43] S. Haouat and R. Chekireb, Mod. Phys. Lett. A. **26**, 2653 (2011)
- [44] C. Bernard, A. Duncun, Ann. Phys. **107**, 201 (1977).
- [45] M.R. Setare, Int. J. Theor. Phys. **43**, 2237 (2004).
- [46] F. Pascoal, C. Farina, Int. J. Theor. Phys. **46** (2007)
- [47] S. Moradi; Int. J. Theor. Phys. **47**, 2808 (2008).
- [48] S. Moradi; Journal of Geometry and Physics **59**, 173 (2009).
- [49] S. P. Kim, H. K. Lee and Y. Yoon; Phys. Rev. D **78**, 105013 (2008)
- [50] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York 1979)